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1. cvičení

1.
Dokažte identitu:

$$\nabla \times (\nabla \times \vec{A}) = \nabla \cdot (\nabla \cdot \vec{A}) - \triangle \vec{A}$$

$$\begin{aligned}
\nabla \times (\nabla \times \vec{A}) &= \nabla \times \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}; \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}; \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\
&= \left(\frac{\partial}{\partial y}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] - \frac{\partial}{\partial z}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]; \frac{\partial}{\partial z}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] - \frac{\partial}{\partial x}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]; \right. \\
&\quad \left. \frac{\partial}{\partial x}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] - \frac{\partial}{\partial y}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] \right)
\end{aligned}$$

$$= \left(\frac{\partial^2 A_y}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial x \partial z}; \frac{\partial^2 A_z}{\partial y \partial z} - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_x}{\partial x \partial y}; \frac{\partial^2 A_x}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_y}{\partial y \partial z} \right) \quad (1)$$

$$\begin{aligned}
\nabla \cdot (\nabla \cdot \vec{A}) - \triangle \vec{A} &= \nabla \cdot \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \\
&\quad - \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}; \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}; \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} \right) \\
&= \left(\frac{\partial}{\partial x}\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right]; \frac{\partial}{\partial y}\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right]; \frac{\partial}{\partial z}\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right] \right) \\
&\quad - \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}; \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}; \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} \right) \\
&= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z}; \frac{\partial^2 A_x}{\partial x \partial y} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y \partial z}; \frac{\partial^2 A_x}{\partial x \partial z} + \frac{\partial^2 A_y}{\partial y \partial z} + \frac{\partial^2 A_z}{\partial z^2} \right) \\
&\quad - \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}; \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}; \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} \right)
\end{aligned}$$

$$= \left(\frac{\partial^2 A_y}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial x \partial z}; \frac{\partial^2 A_z}{\partial y \partial z} - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_x}{\partial x \partial y}; \frac{\partial^2 A_x}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_y}{\partial y \partial z} \right) \quad (2)$$

$$(1) = (2) \Rightarrow \nabla \times (\nabla \times \vec{A}) = \nabla \cdot (\nabla \cdot \vec{A}) - \triangle \vec{A} \qquad \text{c.b.d.}$$

2.

$$\vec{F} = (kx; ky; kz)$$

$$\begin{aligned}
\text{div}\, \vec{F} &= k + k + k = 3k \\
\text{rot}\, \vec{F} &= (0-0; 0-0; 0-0) = \vec{0} \\
\vec{r} &= (k \sin\varphi; k\cos\varphi; k), \quad \varphi \in \langle 0; 2\pi \rangle \\
\mathrm{d}\vec{r} &= (k\cos\varphi; -k\sin\varphi; 0)\mathrm{d}\varphi \\
\vec{F} &= (k^2 \sin\varphi; k^2\cos\varphi; k^2)
\end{aligned}$$

$$\begin{aligned}
W &= \int_C \vec{F}\mathrm{d}\vec{r} = \int_0^{2\pi} (k^2\sin\varphi; k^2\cos\varphi; k^2)\cdot(k\cos\varphi; -k\sin\varphi; 0)\mathrm{d}\varphi = \int_0^{2\pi} (k^3\sin\varphi\cos\varphi - k^3\cos\varphi\sin\varphi + 0)\mathrm{d}\varphi \\
&= \int_0^{2\pi} 0\cdot \mathrm{d}\varphi = 0\,\mathrm{J}
\end{aligned}$$