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4. cvičení

1.
Válcové rozložení náboje:

$$\begin{aligned} r < a &\qquad \varrho = 0 \\ r > a &\qquad \varrho = \frac{k}{r} \end{aligned}$$

Ve válcových souřadnicích je tedy potenciál φ závislý pouze na r a ne na θ a z.
Poissonova rovnice má tedy tvar:

$$\triangle\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) = -\frac{\varrho}{\varepsilon_0}$$

$r < a$

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi_1}{\partial r}\right) &= 0 \\ \frac{\partial}{\partial r}\left(r\frac{\partial\varphi_1}{\partial r}\right) &= 0 \\ r\frac{\partial\varphi_1}{\partial r} &= A \\ \frac{\partial\varphi_1}{\partial r} &= \frac{A}{r} \\ \varphi_1 &= A\ln r + B \end{aligned}$$

$$\begin{aligned} \oint_S \vec{E}_1\,\mathrm{d}\vec{S} &= 0 \\ E_1 &= 0 = -\mathrm{grad}\,\varphi_1 \\ 0 &= -\frac{\partial\varphi_1}{\partial r} = \frac{A}{r} \Rightarrow A = 0 \\ \varphi_1 &= B = \varphi_0 \end{aligned}$$

$r > a$

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi_2}{\partial r}\right) &= -\frac{\varrho}{\varepsilon_0} = -\frac{k}{r\varepsilon_0} \\ \frac{\partial}{\partial r}\left(r\frac{\partial\varphi_2}{\partial r}\right) &= -\frac{k}{\varepsilon_0} \\ r\frac{\partial\varphi_2}{\partial r} &= -\frac{k}{\varepsilon_0}r + C \\ \frac{\partial\varphi_2}{\partial r} &= -\frac{k}{\varepsilon_0} + \frac{C}{r} \\ \varphi_2 &= -\frac{k}{\varepsilon_0}r + C\ln r + D \end{aligned}$$

$$
\begin{aligned}
\oint\limits_S \vec{E}_2 \,\mathrm{d}\vec{S} &= \frac{Q_C}{\varepsilon_0} \\
E_2 2\pi r h &= \frac{\pi(r^2-a^2)\varrho}{\varepsilon_0} = \frac{\pi k h}{\varepsilon_0}\frac{(r^2-a^2}{r} \\
E_2 &= \frac{k}{2\varepsilon_0}\frac{(r^2-a^2)}{r^2} = \frac{k}{2\varepsilon_0}\left(1-\frac{a^2}{r^2}\right) \\
E_2 &= -\mathrm{grad}\,\varphi_2 \\
\frac{k}{2\varepsilon_0}\left(1-\frac{a^2}{r^2}\right) &= \frac{k}{\varepsilon_0} - \frac{C}{r} \\
C &= \frac{k}{2\varepsilon_0}\frac{(r^2+a^2)}{r} \\
\varphi_2 &= -\frac{k}{\varepsilon_0}r + \frac{k}{2\varepsilon_0}\frac{(r^2+a^2)}{r}\ln r + D
\end{aligned}
$$

$$
\begin{aligned}
\varphi_1(a) &= \varphi_2(a) \\
\varphi_0 &= -\frac{k}{\varepsilon_0}a + \frac{k}{2\varepsilon_0}\frac{(a^2+a^2)}{a}\ln a + D \\
\varphi_0 &= \frac{k}{\varepsilon_0}a(\ln a - 1) + D \\
D &= \varphi_0 - \frac{k}{\varepsilon_0}a(\ln a - 1) \\
\varphi_2 &= -\frac{k}{\varepsilon_0}r + \frac{k}{2\varepsilon_0}\frac{(r^2+a^2)}{r}\ln r + \varphi_0 - \frac{k}{\varepsilon_0}a(\ln a - 1) \\
\varphi_2 &= \varphi_0 + \frac{k}{\varepsilon_0}\left(\frac{(r^2+a^2)}{2r}\ln r - r - a(\ln a - 1)\right)
\end{aligned}
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2.
Náboje Q_1 a $-Q_2$ umístěme na pozice $Q_1 = [-d, 0, 0]$ a $-Q_2 = [d, 0, 0]$.
Počítejme potenciál v místě $[x, y, z]$, tedy průvodiče od jednotlivých nábojů jsou $\vec{r}_1 = (x + d, y, z)$ a $\vec{r}_2 = (x - d, y, z)$.
Potenciál spočítáme principem superpozice a položíme rovno nule:

$$\varphi = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{\sqrt{(x+d)^2+y^2+z^2}} + \frac{-Q_2}{\sqrt{(x-d)^2+y^2+z^2}}\right) = 0$$

$$\begin{aligned}
\frac{Q_1}{\sqrt{(x+d)^2+y^2+z^2}} &= \frac{Q_2}{\sqrt{(x-d)^2+y^2+z^2}} \\
Q_1^2 \cdot [(x-d)^2+y^2+z^2] &= Q_2^2 \cdot [(x+d)^2+y^2+z^2] \\
(Q_1^2-Q_2^2)x^2 - 2(Q_1^2+Q_2^2)xd + (Q_1^2-Q_2^2)d^2 + (Q_1^2-Q_2^2)(y^2+z^2) &= 0 \\
x^2 - 2\frac{Q_1^2+Q_2^2}{Q_1^2-Q_2^2}xd + d^2 + y^2 + z^2 &= 0 \\
\left(x - d\sqrt{\frac{Q_1^2+Q_2^2}{Q_1^2-Q_2^2}}\right)^2 - d^2\frac{Q_1^2+Q_2^2}{Q_1^2-Q_2^2} + d^2 + y^2 + z^2 &= 0 \\
\left(x - d\sqrt{\frac{Q_1^2+Q_2^2}{Q_1^2-Q_2^2}}\right)^2 + y^2 + z^2 &= d^2\left(\frac{Q_1^2+Q_2^2}{Q_1^2-Q_2^2} - 1\right)
\end{aligned}$$

Vidíme, že se jedná o rovnici kulové plochy se středem v bodě $\left[d\sqrt{\frac{Q_1^2+Q_2^2}{Q_1^2-Q_2^2}}, 0, 0\right]$ a poloměrem $d\sqrt{\frac{Q_1^2+Q_2^2}{Q_1^2-Q_2^2} - 1}$