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11. Určete osy a vrcholy kuželosečky

$$k\colon \quad x^2 + 2xy + y^2 - 3x - y - 4 = 0.$$

Hlavní čísla:

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = 1 - 2\lambda + \lambda^2 - 1 = \lambda(\lambda - 2) = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = 0$$

Hlavní směry:

$$\lambda_1 = 2\colon \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \end{pmatrix} \Rightarrow \mathbf{u}_1 = (1,1)$$

$$\mathbf{u}_2 = (1,-1)$$

Osy:

$$\begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & -\frac{3}{2} \\ 1 & 1 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$2x_1 + 2x_2 - 2x_3 = 0$$

$$o_1\colon \quad x + y - 1 = 0$$

$$x = 1 - y$$

$$\begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & -\frac{3}{2} \\ 1 & 1 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-x_3 = 0$$

nevlastní přímka není osou

Vrcholy:

$$o_1 \cap k\colon \quad (1-y)^2 + 2(1-y)y + y^2 - 3(1-y) - y - 4 = 0$$

$$2y - 6 = 0$$

$$y = 3 \Rightarrow x = -2$$

$$V = [-2, 3]$$

12. Pomocí transformací kartézských souřadnic určete typ a kanonickou rovnici kuželosečky k. Určete také transformaci kartézských souřadnic, která převede rovnici kuželosečky do kanonického tvaru.
a)

$$k: \quad 4x^2 - 4xy + y^2 - 3x + 4y - 7 = 0$$

$$A = \begin{pmatrix} 4 & -2 & -\frac{3}{2} \\ -2 & 1 & 2 \\ -\frac{3}{2} & 2 & -7 \end{pmatrix} \qquad \begin{array}{l} |A| = -28 + 6 + 6 - \frac{9}{4} - 16 + 28 = -\frac{25}{4} \neq 0 \;\Rightarrow\; \text{regulární} \\ |\bar{A}| = 4 - 4 = 0 \;\Rightarrow\; \text{parabolický typ} \end{array}$$

Typ: parabola
Hlavní čísla:

$$\begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) - 4 = 4 - 5\lambda + \lambda^2 - 4 = \lambda(\lambda - 5) = 0$$

$$\lambda_1 = 5, \quad \lambda_2 = 0$$

Kanonická rovnice kuželosečky k:

$$k: \quad \lambda_1 x'^2 + 2\sqrt{-\frac{|A|}{\lambda_1}}\,y' = 0$$

$$k: \quad 5x'^2 + 2\sqrt{\frac{\frac{25}{4}}{5}}\,y' = 0$$

$$k: \quad 5x'^2 + \sqrt{5}y' = 0$$

Hlavní směry:

$$\lambda_1 = 5: \quad \begin{pmatrix} -1 & -2 \\ -2 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \end{pmatrix} \Rightarrow \mathbf{u}_1 = (2, -1) \Rightarrow \mathbf{e}_1 = \frac{1}{\sqrt{5}}\mathbf{u}_1 = \left(\frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right)$$

$$\mathbf{u}_2 = (1, 2) \quad \Rightarrow \mathbf{e}_2 = \frac{1}{\sqrt{5}}\mathbf{u}_2 = \left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$$

Osa:

$$\begin{pmatrix} 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -2 & -\frac{3}{2} \\ -2 & 1 & 2 \\ -\frac{3}{2} & 2 & -7 \end{pmatrix} o_1: \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$o_1: \quad 10x_1 - 5x_2 - 5x_3 = 0$$

$$o_1: \quad 2x - y - 1 = 0$$

$$y = 2x - 1$$

Vrchol:

$$o_1 \cap k: \quad 4x^2 - 4x(2x-1) + (2x-1)^2 - 3x + 4(2x-1) - 7 = 0$$

$$5x - 10 = 0$$

$$x = 2 \Rightarrow y = 3$$

$$V = [2, 3]$$

Transformační rovnice:

$$\begin{aligned} x &= \frac{2\sqrt{5}}{5}x' + \frac{\sqrt{5}}{5}y' + 2 \\ y &= -\frac{\sqrt{5}}{5}x' + \frac{2\sqrt{5}}{5}y' + 3 \end{aligned}$$

b)

$$k\colon \quad 5x^2 + 8xy + 5y^2 - 18x - 18y + 9 = 0$$

$$A = \begin{pmatrix} 5 & 4 & -9 \\ 4 & 5 & -9 \\ -9 & -9 & 9 \end{pmatrix} \qquad \begin{aligned} &|A| = 225 + 324 + 324 - 405 - 405 - 144 = -81 \neq 0 \ \Rightarrow \ \text{regulární} \\ &|\bar{A}| = 25 - 16 = 9 > 0 \ \Rightarrow \ \text{eliptický typ} \end{aligned}$$

Hlavní čísla:

$$\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 16 = 25 - 10\lambda + \lambda^2 - 16 = \lambda^2 - 10\lambda - 16 = (\lambda - 1)(\lambda - 9) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = 9$$

Kanonická rovnice kuželosečky k:

$$\begin{aligned} k\colon \quad \lambda_1 {x'}^2 + \lambda_2 {y'}^2 + \frac{|A|}{|\bar{A}|} &= 0 \\ k\colon \quad {x'}^2 + 9{y'}^2 - \frac{81}{9} &= 0 \\ k\colon \quad {x'}^2 + 9{y'}^2 - 9 &= 0 \end{aligned}$$

Typ: reálná elipsa

Hlavní směry:

$$\lambda_1 = 1\colon \quad \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \end{pmatrix} \Rightarrow \mathbf{u}_1 = (1, -1) \Rightarrow \mathbf{e}_1 = \frac{1}{\sqrt{2}}\mathbf{u}_1 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$\mathbf{u}_2 = (1, 1) \quad \Rightarrow \mathbf{e}_2 = \frac{1}{\sqrt{2}}\mathbf{u}_2 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

Střed:

$$\left(\begin{array}{ccc|c} 5 & 4 & -9 & 0 \\ 4 & 5 & -9 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 4 & 5 & -9 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 9 & -9 & 0 \end{array}\right) \Rightarrow (1, 1, 1)$$

$S = [1, 1]$

Transformační rovnice:

$$\begin{aligned} x &= \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' + 1 \\ y &= -\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' + 1 \end{aligned}$$

c)

$$k: \quad x^2 - 2xy + y^2 - 12x + 12y - 14 = 0$$

$$A = \begin{pmatrix} 1 & -1 & -6 \\ -1 & 1 & 6 \\ -6 & 6 & -14 \end{pmatrix} \qquad \begin{array}{l} |A| = -14 + 36 + 36 - 36 - 36 + 14 = 0 \;\Rightarrow\; \text{singulární} \\ |\bar{A}| = 1 - 1 = 0 \;\Rightarrow\; \text{parabolický typ} \end{array}$$

Hlavní čísla:

$$\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = 1 - 2\lambda + \lambda^2 - 1 = \lambda(\lambda - 2) = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = 0$$

$\Gamma_1 = a_{11}a_{33} - a_{13}^2 + a_{22}a_{33} - a_{23}^2 = -14 - 36 - 14 - 36 = -100$
Kanonická rovnice kuželosečky k:

$$\begin{aligned} k: \quad \lambda_1 x'^2 + \frac{\Gamma_1}{\lambda_1} &= 0 \\ k: \quad 2x'^2 - \frac{100}{2} &= 0 \\ k: \quad 2x'^2 - 50 &= 0 \\ k: \quad x'^2 - 25 &= 0 \end{aligned}$$

Typ: reálné rovnoběžky
Hlavní směry:

$$\lambda_1 = 2: \quad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \end{pmatrix} \Rightarrow \mathbf{u}_1 = (1, -1) \Rightarrow \mathbf{e}_1 = \frac{1}{\sqrt{2}}\mathbf{u}_1 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$\mathbf{u}_2 = (1, 1) \quad \Rightarrow \mathbf{e}_2 = \frac{1}{\sqrt{2}}\mathbf{u}_2 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

Střed:

$$\left(\begin{array}{ccc|c} 1 & -1 & -6 & 0 \\ -1 & 1 & 6 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & -6 & 0 \end{array}\right) \;\Rightarrow\; \text{např. } (6, 0, 1)$$

$S = [6, 0]$
Transformační rovnice:

$$\begin{aligned} x &= \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' + 6 \\ y &= -\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' \end{aligned}$$